STUDENT NAME: ________________________

TEACHER: ________________________________

HURLSTONE
AGRICULTURAL
HIGH
SCHOOL

PRO PATRIA

2021 | HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK 4

Mathematics Extension 1

Examiners

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General Instructions

- Preparation time – 10 minutes
- Working time – 2 hours
- Scanning and uploading time – 1 hour
- Write using black pen
- Calculators approved by NESA may be used
- A Reference sheet is provided at the back of this paper
- Multiple Choice Answer sheet is provided at the back of this paper
- For questions in Section II, complete on your own paper and show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

SECTION I

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10

Question 1

What is the domain of the function $y = 5\cos^{-1}(4x)$?

A. $-\frac{5}{4} \le x \le \frac{5}{4}$

B. $-\frac{1}{5} \le x \le \frac{1}{5}$

C. $-\frac{1}{4} \le x \le \frac{1}{4}$

D. $-1 \le x \le 1$

Question 2

Which of the following gives the coefficient of x^4 in the expansion of $\left(x-\sqrt{2}\right)^6$?

A. -30

B. 30

C. -60

D 60

Question 3

Let $P(x) = x^2 + bx + c$ where b and c are constants. The zeroes of $P(x)$ are α and $\alpha + 1$. What are the correct expressions for b and c in terms of α ?

A. $b = -(2\alpha + 1)$ and $c = \alpha^2 + \alpha$

B. $b = 2\alpha + 1$ and $c = \alpha^2 + \alpha$

C. $b = \alpha^2 + \alpha$ and $c = -(2\alpha + 1)$

D $b = \alpha^2 + \alpha$ and $c = 2\alpha + 1$

Question 4

The rate of change of the population *(P)* of a small regional centre in NSW can be approximated by the formula $\frac{dP}{dt} = -\frac{1}{10}(P - 10000)$.
Which one of the equations below is a solution to this differential equation?

A. $P = 10000e^{-0.1t}$

B. $P = -10000e^{-0.1t}$

C. $P = 10000 - 2000e^{-0.1t}$

D. $P = 10000 + 2000e^{0.1t}$

Question 5

Which of the following is an expression for $y = \sin x + \sqrt{3}\cos x$ in the form $R\sin(x + \alpha)$?

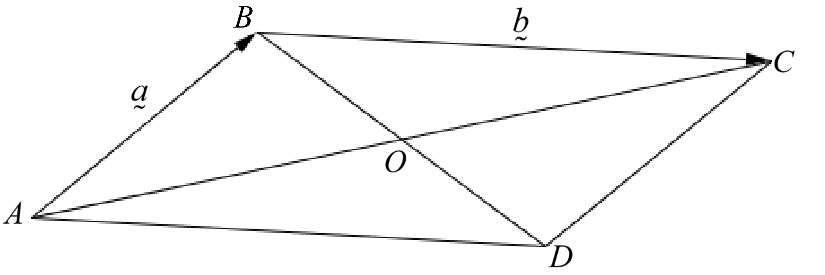
A. $2\sin\left(x + \frac{\pi}{4}\right)$

B. $2\sin\left(x + \frac{\pi}{3}\right)$

C. $2\sin\left(x - \frac{\pi}{4}\right)$

D. $2\sin\left(x - \frac{\pi}{3}\right)$

Question 6



In the parallelogram *ABCD* shown, the point of intersection of the diagonals is *O*.

$\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{BC} = \underset{\sim}{b}$

What is the vector $\overrightarrow{OC}$ equal to?

A. $\frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b})$

B. $\frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

C. $\frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b}$

D. $\frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a}$

Question 7

If $y = \sin^{-1}\frac{a}{x}$, which of the following is an expression for $\frac{dy}{dx}$?

A. $\frac{-a}{x^2\sqrt{x^2 - a^2}}$

B. $\frac{x}{\sqrt{x^2 - a^2}}$

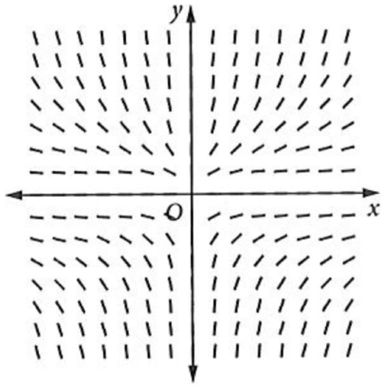
C. $\frac{-x}{\sqrt{x^2 - a^2}}$

D. $\frac{-a}{x\sqrt{x^2 - a^2}}$

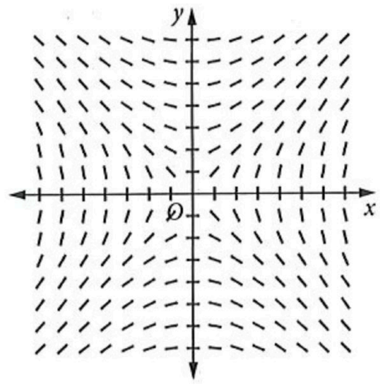
Question 8

Which one of the slope fields below could be used to represent $xy' - y = 0$?

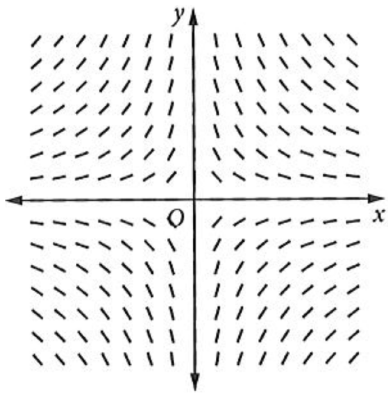
A.



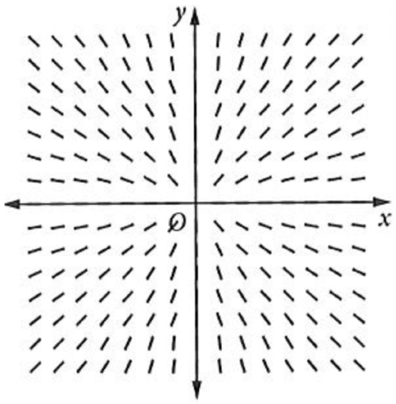
B.



C.



D.



Question 9

Which of the following is a true statement?

A. $\sin 3x \sin 4y = \frac{1}{2}\left[\cos(3x-4y)-\cos(3x+4y)\right]$

B. $\sin 3x \sin 4y = \frac{1}{2}\left[\cos(3x+4y)-\sin(3x-4y)\right]$

C. $\sin 3x \sin 4y = \frac{1}{2}\left[\cos(3x+4y)-\cos(3x-4y)\right]$

D. $\sin 3x \sin 4y = \frac{1}{2}\left[\cos(3x-4y)-\sin(3x+4y)\right]$

Question 10

What is the unit vector in the direction of $\underset{\sim}{a} = -2\underset{\sim}{i} + 5\underset{\sim}{j}$?

A. $\frac{1}{7}\left(-2\underset{\sim}{i} + 5\underset{\sim}{j}\right)$

B. $\frac{1}{29}\left(-2\underset{\sim}{i} + 5\underset{\sim}{j}\right)$

C. $\frac{1}{\sqrt{29}}\left(-2\underset{\sim}{i} + 5\underset{\sim}{j}\right)$

D. $\frac{1}{\sqrt{21}}\left(-2\underset{\sim}{i} + 5\underset{\sim}{j}\right)$

End of Section I

SECTION II

60 marks

Attempt Questions 11 – 14

Allow about 1hour and 45 minutes for this section.

Answer each question on your own paper.

For questions **in Section II**, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a new Writing Booklet

(a) Four politicians and four activists are seated alternately at a round table. **1**

(i.e. alternating so that no two politicians or two activists sit next to each other)

In how many ways can this be done?

(b) The letters of the word CONSIDER are rearranged at random. What is the probability that an arrangement begins with 'SI' ? **1**

(c) A selection from the integers from 1 to 8 is made. How many numbers need to be selected from this group in order to be certain that the selection contains a pair of numbers that add up to 10? **1**

Question 11 continued on Page 8

(d) Two boats on a lake start sailing at the same time.

Boat A moves on a course given by: $x = \frac{t}{2}$; $y = t + 1$.

Boat B moves on a course given by: $x = t - 2$: $y = 9 - 2t$.

where t measures the time elapsed in hours from when the boats started.

(i) Find the Cartesian equation for the course of each boat and show that the courses intersect at $(1, 3)$. **2**

(ii) Do the boats collide? Justify your answer supported by calculations. **1**

(e) Find the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ if $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. **2**

(f) Find $\underset{\sim}{d}$ perpendicular to $\underset{\sim}{c} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$ with a magnitude of 10. **3**

(g) ΔABC is a triangle with $\overrightarrow{AB} = \underset{\sim}{c}$ and $\overrightarrow{AC} = \underset{\sim}{b}$.

D and E are the midpoints of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively.

F is a point on $\overrightarrow{BC}$ such that $\overrightarrow{FC} = 2 \times \overrightarrow{BF}$

(i) Express the vectors $\overrightarrow{BC}$ and $\overrightarrow{DE}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$. **1**

(ii) What geometric property of a triangle is demonstrated by the answers to (i)? **1**

(iii) Express $\overrightarrow{AF}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$. **2**

End of Question 11

Question 12 (15 marks) Use a new Writing Booklet

(a) Consider the function $f(x) = x^2 - 4x + 6$.

(i) Explain why the domain of $f(x)$ must be restricted if $f(x)$ is to have an inverse function. **1**

(ii) Given that the domain of $f(x)$ is restricted to $x \le 2$, find an expression for $f^{-1}(x)$. **2**

(b) Differentiate $e^x \tan^{-1}(x)$. **2**

(c) Find $\displaystyle\int_0^{\pi} \frac{4}{\sqrt{16-x^2}}dx$. **2**

(d) Find $\displaystyle\int 2\cos^2(x)\,dx$. **2**

(e) Using the substitution $u = x^2 - 2$, or otherwise, find the exact value of $\displaystyle\int_2^3 \frac{x}{\sqrt{x^2-2}}dx$. **3**

(f) Prove by mathematical induction that $4^n + 14$ is divisible by 6 for all positive integers n where $n \ge 1$. **3**

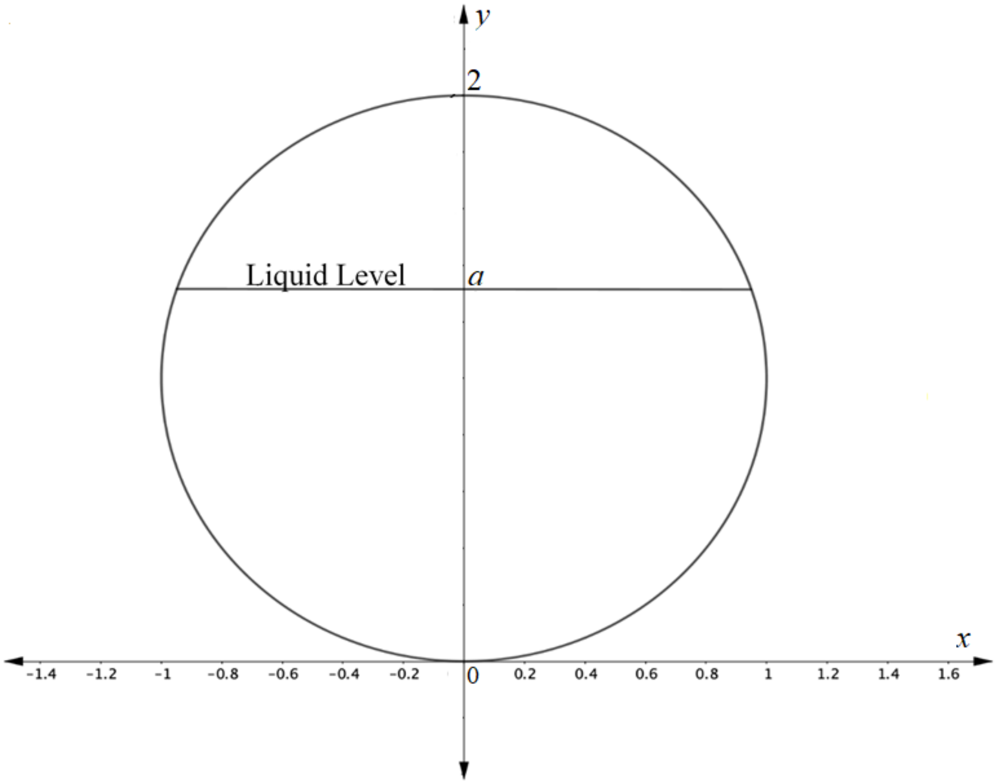
End of Question 12

Question 13 (15 marks) Use a new Writing Booklet

(a) A spherical tank used to transport an industrial liquid has a radius of 1 metre.

The volume of liquid in the tank for a depth of a metres can be found by calculating the volume of a suitable solid of revolution using integration methods.

The diagram below shows the cross section of the tank through its centre drawn on a cartesian plane.



(i) Show that the volume of liquid, of depth a, in the tank is given by: **3**

$$V = \frac{\pi a^2}{3}(3 - a)$$

where a is measured in metres.

(ii) The tank in (i) above is being filled at a constant rate of $\frac{\pi}{10}$ m^3/min.

(α) Find the rate of change of volume (V) with respect to depth (a). **1**

(β) At what rate is the depth of the liquid increasing when the depth is 0.4 m? **2**

Question 13 continued on Page 11

(b) The region bounded by the curve $y = e^{-x}$, the coordinate axes and the line $x = b$, $b > 0$, is rotated about the x-axis.

(i) Find the volume of the solid of revolution generated in terms of b. **2**

(ii) Find the limit of the volume of this solid as $b \to \infty$. **1**

(c) Solve the differential equation $\dfrac{dy}{dx} = \cos^2 y$ given that $y = \dfrac{\pi}{4}$ when $x = 0$. **2**

(d) The formula for the height of a tree, H, after t years can be found by solving the differential equation $\dfrac{dH}{dt} = 0.03H\left(45 - H\right)$, given the tree is 0.3 metres high when it is planted.

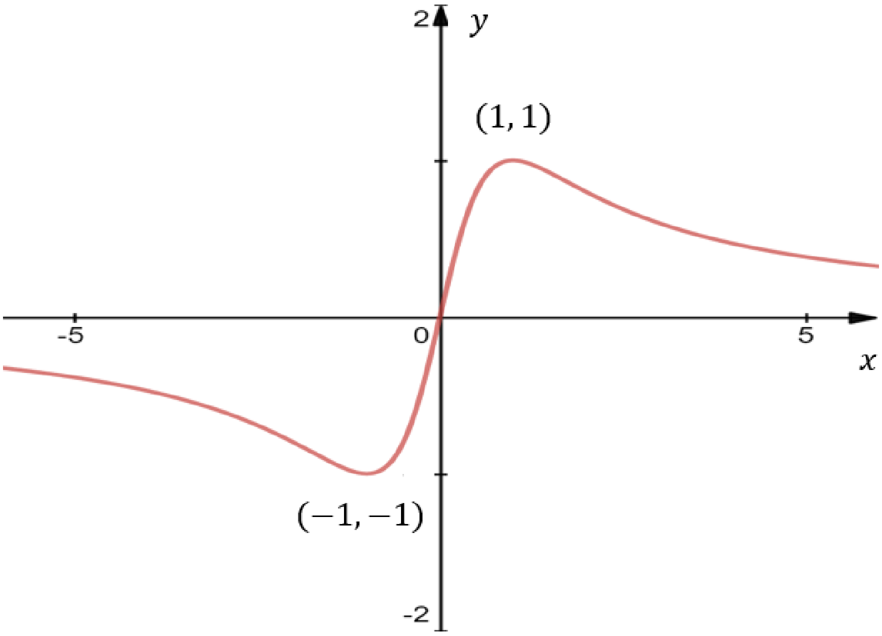
(i) Show that the expression $\dfrac{1}{H\left(45-H\right)} = \dfrac{1}{H} + \dfrac{1}{45-H}$. **1**

(ii) Solve the differential equation to find the formula for the height of the tree. **3**

End of Question 13

Question 14 (15 marks) Use a new Writing Booklet

(a) The graph of the function $y = \dfrac{2x}{\left(x^2+1\right)}$ is shown below. **1**



Sketch the graph of $y = \dfrac{2|x|}{\left(x^2+1\right)}$. Label all important features.

(b) By dividing the polynomial $f(x) = 2x^4 - 10x^3 + 12x^2 + 2x - 3$ by **2**

$g(x) = x^2 - 3x + 1$ write $f(x)$ in the form shown below:

$f(x) = g(x)q(x) + r(x)$ where $q(x)$ and $r(x)$ are polynomials

and $r(x)$ has degree less than 2.

Question 14 continued on Page 13

(c) Given that $(\sin\alpha - \cos\alpha)^2 = 1 - \sin 2\alpha$, What is the exact value of $\sin 75^o - \cos 75^o$? **2**

Leave your answer with a rational denominator.

(d) (i) If $\sin(x+\theta) = A\sin(x-\theta)$, prove that $(A-1)\tan x = (A+1)\tan\theta$. **3**

(ii) Hence solve $\sin(x+40^o) = 3\sin(x-40^o)$, for $0^o \le x \le 360^o$. **2**

(e) (i) Prove that $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$. **2**

(ii) Given A, B and C are acute angles of a triangle and $\dfrac{\tan A}{5} = \dfrac{\tan B}{6} = \dfrac{\tan C}{7} = k$. **3**

Hence or otherwise show that $k = \sqrt{\dfrac{3}{35}}$ is the only solution to $\tan(A+B+C) = 0$.

End of Paper

Year 12 2021 HSC Assessment Task 4 Marking Guidelines

Answers to Multiple Choice questions

Question 1:

$$y = 5\cos^{-1}(4x)$$
$$-1 \le 4x \le 1$$
$$-\frac{1}{4} \le x \le \frac{1}{4}$$

$\therefore$ **Answer C**

Question 2:

The term that contains x^4 in the expansion of $\left(x-\sqrt{2}\right)^6$ is $6C_4 x^4\left(-\sqrt{2}\right)^2$

Coefficient of $x^4 = 6C_4\left(-\sqrt{2}\right)^2 = 15\times 2 = 30$

$\therefore$ **Answer B**

Question 3:

Since the zeroes of $P(x)$ are α and $\alpha+1$

Sum of roots: $\alpha+\alpha+1 = -\frac{b}{1}$

$2\alpha+1 = -b$

$\therefore b = -(2\alpha+1)$

$\therefore$ **Answer A**

Product of roots

$\alpha\times(\alpha+1) = \frac{c}{1}$

$\therefore c = \alpha^2+\alpha$

Question 4:

Upon differentiating

$$P = 10000 - 2000e^{-0.1t}$$
$$\frac{dP}{dt} = 200e^{-0.1t}$$
$$= \frac{1}{10}\left(2000e^{-0.1t}\right)$$

but, $2000e^{-0.1t} = 10000 - P$

$$\therefore \frac{dP}{dt} = \frac{1}{10}(10000-P)$$

$= -\frac{1}{10}(P-10000)$, the required differential equation

$\therefore$ **Answer C**

Question 5:

$$R = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\tan\alpha = \frac{\sqrt{3}}{1}, \quad \alpha = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

$$\therefore \sin x + \sqrt{3}\cos x = 2\sin\left(x + \frac{\pi}{3}\right)$$

$\therefore$ **Answer B**

Question 6:

Diagonals of a parallelogram bisect each other. $\therefore \overrightarrow{OC} = \frac{1}{2}\overrightarrow{AC}$

$\therefore$ **Answer B**

Question 7:

$$y = \sin^{-1}\frac{a}{x}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \left(\frac{a}{x}\right)^2}} \times -\frac{a}{x^2}$$

$$= \frac{-a}{x\sqrt{x^2 - a^2}}$$

$\therefore$ **Answer D**

Question 8:

The differential equation $xy' - y = 0$ can be rearranged to give $y' = \frac{y}{x}$.

When $x > 0$ and $y > 0$, $y' > 0$. This rules out option C as gradients are negative in the first quadrant.

Also, when $x < 0$ and $y < 0$, $y' > 0$. This rules out A as gradients are negative in the third quadrant.

Lastly, when $x = 0$, gradients will be undefined. This leaves D as the correct answer as tangents are vertical when $x = 0$ and B has horizontal tangents when $x = 0$.

$\therefore$ **Answer D**

Question 9:

From refernce sheet using $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$\sin 3x \sin 4y = \frac{1}{2}\left[\cos(3x-4y) - \cos(3x+4y)\right]$ is a true statement

$\therefore$ **Answer A**

Question 10:

All 4 responses feature the same vector ***a*** in brackets.
To get unit vector, it's divided by its length. Use Pythagoras to do so.
$\therefore$ **Answer C**

Year 12 Mathematics Extension 1 Assessment Task 4 2021

Question No. 11 Solutions and Marking Guidelines

Outcomes Addressed in this Question

ME 11-1: uses algebraic and graphical concepts in the modelling and solving of problems involving functions
ME 11-5: uses concepts of permutations and combinations to solve problems involving counting or ordering
ME 12-2: Applies concepts and techniques involving vectors to solve problems

Outcome	Solutions	Marking Guidelines
ME 11-5	**(a)** $3! \times 4! = 144$	**(a)** **Award 1:** Correct answer.
	(b) $\frac{6!}{8!}$ or $\frac{1}{8}\times\frac{1}{7}=\frac{1}{56}$	**(b)** **Award 1:** Correct answer.
	(c) 5 pigeonholes: [1], [5], [2,8], [3,7], [4,6] Therefore 6 values need to be selected to be sure of a pair that add to 10.	**(c)** **Award 1:** Correct answer.
ME 11-1	**(d)** **(i)** Boat A: $y=2x+1$ Boat B: $y=5-2x$ Substitute x = 1 to RHS to show that LHS = 3 = y Boat A: $2(1)+1=3$ Boat B: $5-2(1)=3$ Therefore the point of intersection is (1, 3).	**(d)** **(i) Award 2:** Correct solution **Award 1** Both equations correct **Award 1** Justification of point of intersection.
	(ii) Boat A is at (1, 3) when t=2, i.e. after 2 hours. Boat B is at (1, 3) when t=3, i.e. after 3 hours. Therefore they do not collide.	**(ii) Award 1:** Correct answer with justification.
ME 12-2	**(e)** Projection = $\frac{1\times2+2\times2}{(2\times2)^2}\times\begin{pmatrix}2\\2\end{pmatrix}=\begin{pmatrix}1\frac{1}{2}\\1\frac{1}{2}\end{pmatrix}$	**(e)** **Award 2** Correct solution. **Award 1** At least one correct dot product, or equivalent.
	(f) Let $\underset{\sim}{d}=x\underset{\sim}{i}+y\underset{\sim}{j} \rightarrow$ dot product: $4x-3y=0$ $\lvert d\rvert=10 \rightarrow \quad x^2+y^2=100$ $\rightarrow x=6,-6 \quad y=8,-8$ $\therefore d=\pm\left(6\underset{\sim}{i}+8\underset{\sim}{j}\right)$ Alternately, you could find the 2 vectors in the form $\underset{\sim}{d}=k\begin{pmatrix}3\\4\end{pmatrix}$ where $\lvert d\rvert^2=100$	**(f)** **Award 3:** Correct solution with working. **Award 2:** Error in working, or failure to find both solutions. **Award 1:** Some significant progress.
	(g) **(i)** $\overrightarrow{BC}=\underset{\sim}{b}-\underset{\sim}{c}$ $\overrightarrow{DE}=\frac{1}{2}\left(\underset{\sim}{b}-\underset{\sim}{c}\right)$	**(g)** **(i) Award 1** both vectors correct.
	(ii) The interval joining the midpoints of 2 sides of s triangle is parallel to and half the length of the third side.	**(ii) Award 1:** Both properties mentioned

	(iii) $\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BF} = \underset{\sim}{c} + \frac{1}{3}\overrightarrow{BC}$ $= \frac{1}{3}\underset{\sim}{b} + \frac{2}{3}\underset{\sim}{c}$	**(iii) Award 2:** Correct solution simplified to one component of ***b*** and ***c***. **Award 1:** Some significant progress.

Year 12	Mathematics Extension 1 2021	TASK 4
Question No. 12	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

ME12-1 – applies techniques involving proof or calculus to model and solve problems

ME11-1 – uses algebraic and graphical concepts in the modelling and solving of problems involving functions and their inverses

Part / Outcome	Solutions	Marking Guidelines
(a)(i) ME11-1	$f(x)=x^2-4x+6$ is a parabola, and **fails the horizontal line test**. So, the inverse of $f(x)$ over its entire domain is not a function.	**1 mark** – Correct solution (explains using the horizontal line test, or equivalent merit)
(a)(ii) ME11-1	Complete the square to express $f(x)$ in vertex form: $f(x)=x^2-4x+6 \quad (x\le 2)$ $=(x-2)^2+2$ Swap x and y, and make y the subject: $x=(y-2)^2+2 \quad (y\le 2)$ $x-2=(y-2)^2$ $y-2=-\sqrt{x-2} \quad \left(\text{discard } +\sqrt{x-2} \text{ as } y\le 2\right)$ $y=-\sqrt{x-2}+2$ $f^{-1}(x)=-\sqrt{x-2}+2$	**2 marks** – Correct solution **1 mark** – Substantially correct *(swaps x and y, or equivalent merit)*
(b) ME12-1	$\frac{d}{dx}\left(e^x\tan^{-1}x\right)=u'v+v'u$ $=e^x\tan^{-1}x+\frac{e^x}{1+x^2}$	**2 marks** – Correct solution **1 mark** – Substantially correct *(uses product rule)*
(c) ME12-1	$\int_0^{\pi}\frac{4}{\sqrt{16-x^2}}\,dx=4\left[\sin^{-1}\left(\frac{x}{4}\right)\right]_0^{\pi}$ $=4\left[\sin^{-1}\left(\frac{\pi}{4}\right)-\sin^{-1}(0)\right]$ $=4\sin^{-1}\left(\frac{\pi}{4}\right) \quad (\approx 3.61)$	**2 marks** – Correct solution **1 mark** – significant progress towards correct solution *(eg finds correct integral)* *** see note below about this question

	Question 12 continued…	
(d) **ME12-1**	$$\int 2\cos^2 x\,dx = 2\int \frac{1}{2}(1+\cos 2x)\,dx$$ $$= x + \frac{1}{2}\sin 2x + C$$	**2 marks** – Correct solution **1 mark** – significant progress towards correct solution *(eg uses double angle formula to simplify integral)*
(e) **ME12-1**	$$u = x^2 - 2$$ $$du = 2x\,dx$$ $$x = 3 \Rightarrow u = 7$$ $$x = 2 \Rightarrow u = 2$$ $$\int_2^3 \frac{x}{\sqrt{x^2-2}}\,dx = \frac{1}{2}\int_2^3 \frac{2x}{\sqrt{x^2-2}}\,dx$$ $$= \frac{1}{2}\int_2^7 \frac{du}{\sqrt{u}} = \frac{1}{2}\int_2^7 u^{-\frac{1}{2}}\,du$$ $$= \left[u^{\frac{1}{2}}\right]_2^7$$ $$= \sqrt{7} - \sqrt{2}$$	**3 marks** – Correct solution **2 marks** – significant progress towards correct solution **1 mark** – partial progress towards correct solution
(f) **ME12-1**	$4^n + 14$ divisible by 6 for $n \geq 1$? Show true for $n = 1$ $4^1 + 14 = 18 = 6(3)$ $\therefore$ true for $n = 1$ Assume true for $n = k$ ie $4^k + 14 = 6p$ $(p \text{ is integer})$ $4^k = 6p - 14$ Prove true for $n = k+1$ ie $4^{k+1} + 14 = 6q$ $(q \text{ is integer})$ $\text{LHS} = 4^{k+1} + 14$ $= 4(4^k) + 14$ $= 4(6p - 14) + 14$ (from assumption) $= 24p - 56 + 14$ $= 24p - 42$ $= 6(4p - 7)$ $= 6q$ $(q \text{ is integer as } 4p - 7 \text{ is an integer})$ $\therefore$ true by mathematical induction	**3 marks** – Correct solution **2 marks** – significant progress towards correct solution **1 mark** – partial progress towards correct solution

	***Please note: (c) Many candidates demonstrated a lack of awareness that $\frac{\pi}{4}$ is not an angle in this question $\sin^{-1}\left(\frac{\pi}{4}\right) \neq \frac{1}{\sqrt{2}}$ … detail is important Similarly: (e) Some candidates did not change the x limits at the substitution stage. When the substitution is made, the limits should also change to be in terms of u. Even if later changing back to be in terms of x. Detail is important.	

Outcomes Addressed in this Question

MA12-7 applies the concepts and techniques of indefinite and definite integrals in the solution of problems

Outcome	Solutions	Marking Guidelines
ME12-7	**(a) (i)** Equation of circular cross section: $x^2+(y-1)^2=1$ $x^2=1-(y-1)^2$ $=1-y^2+2y-1$ $=2y-y^2$ Volume generated by rotating the curve about the y-axis between 0 and a (ie. the depth of the liquid in the tank) $V=\pi\int_a^b x^2\,dy$ $=\pi\int_0^a 2y-y^2\,dy$ $=\pi\left[y^2-\frac{y^3}{3}\right]_0^a$ $=\pi\left(a^2-\frac{a^3}{3}\right)$ $=\pi\left(\frac{3a^2-a^3}{3}\right)$ $=\frac{\pi a^2}{3}(3-a)$ as required	**3 marks** Correct solution showing equation of circular cross section, primitive function and substitution leading to required expression. **2 marks** Substantial progress towards correct solution. **1 mark** Some progress towards correct solution showing knowledge of the process of finding the volume of a solid of revolution.
ME12-7	**(ii) (α)** $V=\frac{\pi a^2}{3}(3-a)$ $=\pi a^2-\frac{\pi a^3}{3}$ $\frac{dV}{da}=2\pi a-\pi a^2$ $=\pi a(2-a)$	**1 mark** Correct answer.
ME12-7	(β) Given $\frac{dV}{dt}=\frac{\pi}{10}$ Rate of change of depth with respect to time $\frac{da}{dt}=\frac{da}{dV}.\frac{dV}{dt}$ $=\frac{1}{\pi a(2-a)}.\frac{\pi}{10}$ $=\frac{1}{10a(2-a)}$ When depth of liquid $a=0.4$ $\frac{da}{dt}=\frac{1}{10\times0.4(2-0.4)}$ $=\frac{1}{4\times1.6}$ $=\frac{5}{32}$ m / min	**2 marks** Correct solution making use of the chain rule. **1 mark** Substantial progress towards correct solution.

ME12-7	**(b) (i)** $y = e^{-x}$ $y^2 = e^{-2x}$ $V = \pi \int_a^b y^2 \, dx$ $= \pi \int_0^b e^{-2x} \, dx$ $= \pi \left[-\frac{1}{2} e^{-2x} \right]_0^b$ $= \pi \left[\left(-\frac{1}{2} e^{-2b} \right) - \left(-\frac{1}{2} e^0 \right) \right]$ $= \pi \left(-\frac{1}{2} e^{-2b} + \frac{1}{2} \right)$ $= \frac{\pi}{2} \left(1 - e^{-2b} \right)$ units^3	**2 marks** Correct solution. **1 mark** Substantial progress towards correct solution.
ME12-7	**(ii)** $\lim_{b \to \infty} \frac{\pi}{2} \left(1 - e^{-2b} \right) = \frac{\pi}{2} (1 - 0)$ since $\lim_{x \to \infty} \left(e^{-x} \right) = 0$ $= \frac{\pi}{2}$ units^3	**1 mark** Correct evaluation of the limit.
ME12-7	**(c)** $\frac{dy}{dx} = \cos^2 y$ $\frac{dx}{dy} = \frac{1}{\cos^2 y}$ $= \sec^2 y$ $x = \tan y + c$ When $y = \frac{\pi}{4}, x = 0$ $\therefore 0 = \tan \frac{\pi}{4} + c$ $c = -\tan \frac{\pi}{4}$ $= -1$ $\therefore x = \tan y - 1$ $x + 1 = \tan y$ $y = \tan^{-1}(x+1)$	**2 marks** Correct solution. **1 mark** Substantial progress towards correct solution.
ME12-7	**(d) (i)**	**1 mark** Correct solution

ME12-7	$$\begin{aligned}\text{RHS} &= \frac{1}{H}+\frac{1}{45-H}\\ &= \frac{(45-H)}{H(45-H)}+\frac{H}{H(45-H)}\\ &= \frac{45-H+H}{H(45-H)}\\ &= \frac{45}{H(45-H)}\end{aligned}$$ OR $$\frac{1}{H(45-H)}=\frac{1}{45}\left(\frac{1}{H}+\frac{1}{45-H}\right)$$ **(ii)** $$\begin{aligned}\frac{dH}{dt} &= 0.03H(45-H)\\ \frac{dt}{dH} &= \frac{1}{0.03H(45-H)}\\ &= \frac{1}{0.03\times 45}\left(\frac{1}{H}+\frac{1}{45-H}\right)\\ \int\frac{dt}{dH}dH &= \frac{1}{1.35}\int\frac{1}{H}+\frac{1}{45-H}dH\\ t &= \frac{1}{1.35}\left[\ln\lvert H\rvert-\ln\lvert 45-H\rvert\right]+c\\ t-c &= \frac{1}{1.35}\ln\left\lvert\frac{H}{45-H}\right\rvert\\ 1.35(t-c) &= \ln\left\lvert\frac{H}{45-H}\right\rvert\\ e^{1.35(t-c)} &= \frac{H}{45-H}, 0\le H\le 45\\ e^{1.35t}\times e^{-1.35c} &= \frac{H}{45-H}\\ Ae^{1.35t} &= \frac{H}{45-H}, \text{ where } A=e^{-1.35c}\end{aligned}$$ When $t=0, H=0.3$ $$\begin{aligned}A &= \frac{0.3}{45-0.3}\\ &= \frac{1}{149}\\ \frac{1}{149}e^{1.35t} &= \frac{H}{45-H}\\ e^{1.35t} &= \frac{149H}{(45-H)}\\ (45-H)e^{1.35t} &= 149H\\ 45e^{1.35t} &= 149H+e^{1.35t}H\\ &= H\left(149+e^{1.35t}\right)\\ H &= \frac{45e^{1.35t}}{149+e^{1.35t}}\\ &= \frac{45}{149e^{-1.35t}+1}\\ &= \frac{45}{1+149e^{-1.35t}}\end{aligned}$$	**3 marks** Correct solution showing all required steps. **2 marks** Substantial progress towards correct solution, including showing correct use of equality shown to be true in part (i). **1 mark** Some progress towards correct solution.

Year 12 Mathematics Extension 1 Assessment Task 4 2021

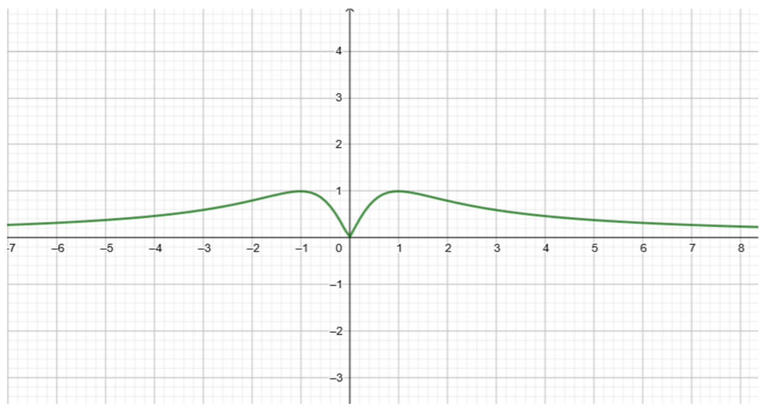
Question No. 14 — Solutions and Marking Guidelines

Outcomes Addressed in this Question

ME11-1 : uses algebraic and graphical concepts in the modelling and solving of problems involving functions and their inverses

ME11-2: manipulates algebraic expressions and graphical functions to solve problems

ME12-3: applies advanced concepts and techniques in simplifying expressions involving compound angles and solving trigonometric equations

Outcome	Solutions	Marking Guidelines
ME11-1	**(a)** 	**Award 1** mark for the correct graph.
ME11-2	**(b)** $\begin{array}{r} 2x^2-4x-2 \\ x^2-3x+1\overline{)2x^4-10x^3+12x^2+2x-3} \\ \underline{2x^4-6x^3+2x^2} \\ 0-4x^3+10x^2+2x-3 \\ \underline{-4x^3+12x^2-4x} \\ 0-2x^2+6x-3 \\ \underline{-2x^2+6x-2} \\ -1 \end{array}$ $\therefore 2x^4-10x^3+12x^2+2x-3=\left[x^2-3x+1\right]\left[2x^2-4x-2\right]-1$	**Award 2** marks for the correct solution. **Award 1** mark for substantial progress towards the solution
ME12-3	**(c)** Using $(\sin\alpha-\cos\alpha)^2=1-\sin 2\alpha$ (given) $(\sin 75^o-\cos 75^o)^2=1-\sin(2\times 75^o)$ $=1-\sin 150^o$ $=1-\sin(180^o-30^o)=1-\sin 30^o=1-\frac{1}{2}=\frac{1}{2}$ $\sin 75^o-\cos 75^o=\pm\sqrt{\frac{1}{2}}=\pm\frac{1}{\sqrt{2}}$ $\because \sin x\geq\cos x$ for $45^o\leq x\leq 90^o$ $\therefore \sin 75^o-\cos 75^o=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}$	**Award 2** marks for the correct solution. **Award 1** mark for substantial progress towards the solution

ME12-3	**(d)(i)** Using $\sin(x+\theta)=A\sin(x-\theta)$ $\sin x\cos\theta+\cos x\sin\theta=A[\sin x\cos\theta-\cos x\sin\theta]$ $\sin x\cos\theta+\cos x\sin\theta=A\sin x\cos\theta-A\cos x\sin\theta$ $A\cos x\sin\theta+\cos x\sin\theta=A\sin x\cos\theta-\sin x\cos\theta$ $(A+1)\cos x\sin\theta=(A-1)\sin x\cos\theta$ divide both sides by $\cos x\cos\theta$ $\frac{(A+1)\cos x\sin\theta}{\cos x\cos\theta}=\frac{(A-1)\sin x\cos\theta}{\cos x\cos\theta}$ $\frac{(A+1)\sin\theta}{\cos\theta}=\frac{(A-1)\sin x}{\cos x}$ $(A+1)\tan\theta=(A-1)\tan x$ $\therefore(A-1)\tan x=(A+1)\tan\theta$ **(d)(ii)** $\sin(x+40^o)=3\sin(x-40^o)$ using part (i) $(3-1)\tan x=(3+1)\tan 40^o$ $2\tan x=4\tan 40^o$ $\tan x=2\tan 40^o$ $x=\tan^{-1}(2\tan 40^o), 180^o+\tan^{-1}(2\tan 40^o)$ $x=59.21^o, 239.21^o$ (correct to two decimal places)	**Award 3** marks for the correct solution. **Award 2** mark for substantial progress towards the correct solution. **Award 1** mark for some progress towards the correct solution. **Award 2** marks for the correct solution. **Award 1** mark for substantial progress towards the solution
ME12-3	**(e) (i)** $\tan(A+B+C)=\tan[A+(B+C)]=\frac{\tan A+\tan(B+C)}{1-\tan A\tan(B+C)}$ $=\frac{\tan A+\frac{\tan B+\tan C}{1-\tan B\tan C}}{1-\tan A\left[\frac{\tan B+\tan C}{1-\tan B\tan C}\right]}$ $=\frac{\frac{\tan A(1-\tan B\tan C)+\tan B+\tan C}{1-\tan B\tan C}}{\frac{1-\tan B\tan C-\tan A(\tan B+\tan C)}{1-\tan B\tan C}}$ $=\frac{\tan A-\tan A\tan B\tan C+\tan B+\tan C}{1-\tan B\tan C-\tan A\tan B-\tan A\tan C}$ $=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$	**Award 2** marks for the correct solution. **Award 1** mark for substantial progress towards the solution

ME12-3	**(e) (ii)** In $\Delta ABC,\ A+B+C=180^o$ $\therefore \tan\left(A+B+C\right)=\tan 180^o=0$ using part (i) $\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}=0$ $\tan A+\tan B+\tan C-\tan A\tan B\tan C=0$ $\tan A+\tan B+\tan C=\tan A\tan B\tan C$ ----------$\boxed{A}$ Given $\dfrac{\tan A}{5}=\dfrac{\tan B}{6}=\dfrac{\tan C}{7}=k$ $\therefore \tan A=5k, \tan B=6k$ and $\tan C=7k$ using $\boxed{A}$ $5k+6k+7k=5k\times 6k\times 7k$ $18k=210k^3$ $210k^3-18k=0$ $6k\left(35k^2-3\right)=0$ $6k=0, k^2=\dfrac{3}{35}$ $\therefore k=0, k=\pm\sqrt{\dfrac{3}{35}}$ The only valid value is $k=+\sqrt{\dfrac{3}{35}}$ since $\tan A>0,\ \tan B>0$ and $\tan C>0$ and *A,B* and *C* are acute.	**Award 3** marks for the correct solution. **Award 2** mark for substantial progress towards the correct solution. **Award 1** mark for some progress towards the correct solution.
ME12-3		